Convergence to equilibrium for smectic-A liquid crystals in 3D domains

F. Guillén-González,

Depto. EDAN & IMUS, Universidad de Sevilla. Spain.

AIMS, 07-11 July 2014, Madrid.

In collaboration: B. Climent-Ezquerra (Univ. Sevilla),

. [NA-TMA'14].

- **Liquid crystals (LC)** exhibit properties between liquids and solids. A **LC** can behave like a liquid (macroscopically), but their molecules have a preferential **orientation** (microscopically), due to **elasticity** effects (**anisotropic liquids**).
- The time-dynamics interaction between **macroscopic** and **microscopic** is modeled by nonlinear parabolic PDE + **gradient flow**, involving:
  - Macroscopic: fluid dynamics (Navier-Stokes)
  - Microscopic: **order parameter**
- Different phases in LC, for instance:
  - **nematic** phases, with orientational order of molecules,
  - **smectic** phases, with moreover positional order (arranged in layers).

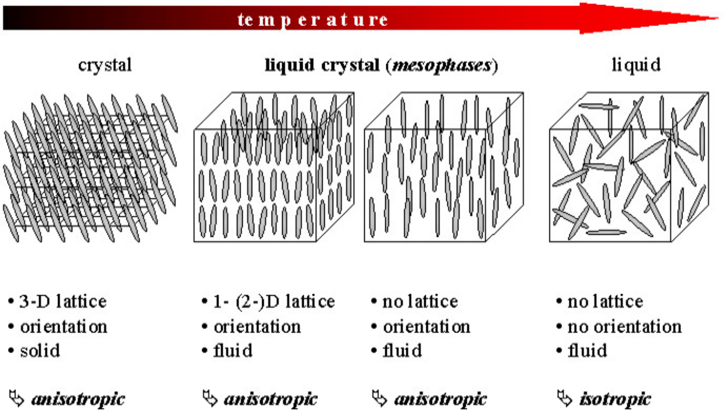
Liquid Crystals

- **Liquid crystals (LC)** exhibit properties between liquids and solids. A **LC** can behave like a liquid (macroscopically), but their molecules have a preferential **orientation** (microscopically), due to **elasticity** effects (**anisotropic liquids**).
- The time-dynamics interaction between **macroscopic** and **microscopic** is modeled by nonlinear parabolic PDE + **gradient flow**, involving:
  - Macroscopic: fluid dynamics (Navier-Stokes)
  - Microscopic: **order parameter**
- Different phases in LC, for instance:
  - **nematic** phases, with orientational order of molecules,
  - **smectic** phases, with moreover positional order (arranged in layers).

Liquid Crystals

- **Liquid crystals (LC)** exhibit properties between liquids and solids. A **LC** can behave like a liquid (macroscopically), but their molecules have a preferential **orientation** (microscopically), due to **elasticity** effects (**anisotropic liquids**).
- The time-dynamics interaction between **macroscopic** and **microscopic** is modeled by nonlinear parabolic PDE + **gradient flow**, involving:
  - Macroscopic: fluid dynamics (Navier-Stokes)
  - Microscopic: **order parameter**
- Different phases in LC, for instance:
  - **nematic** phases, with orientational order of molecules,
  - **smectic** phases, with moreover positional order (arranged in layers).

thermotropic liquid crystals



Static Osseen-Frank's theory for nematic LC

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$): domain filled by the LC, with boundary $\partial\Omega$
- Equilibrium states: minimum of a free energy
- **Oseen-Frank** free energy derive to the Static (Minimization) problem:

$$\min_{|\boldsymbol{d}|=1} E_{ela}(\boldsymbol{d})$$

where E_{ela} is the **elastic** energy functional:

$$E_{ela}(\boldsymbol{d}) = \int_\Omega \left(\frac{k_1}{2}(\nabla \cdot \boldsymbol{d})^2 + \frac{k_2}{2}(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 + \frac{k_3}{2}|\boldsymbol{d} \times (\nabla \times \boldsymbol{d})|^2 \right)$$

$k_i > 0$ elastic constants.

Remark: Problem with convex functional but non-convex constraints.

Static Osseen-Frank's theory for nematic LC

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$): domain filled by the LC, with boundary $\partial\Omega$
- Equilibrium states: minimum of a free energy
- **Oseen-Frank** free energy derive to the Static (Minimization) problem:

$$\min_{|\boldsymbol{d}|=1} E_{ela}(\boldsymbol{d})$$

where E_{ela} is the **elastic** energy functional:

$$E_{ela}(\boldsymbol{d}) = \int_\Omega \left(\frac{k_1}{2}(\nabla \cdot \boldsymbol{d})^2 + \frac{k_2}{2}(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 + \frac{k_3}{2}|\boldsymbol{d} \times (\nabla \times \boldsymbol{d})|^2 \right)$$

$k_i > 0$ elastic constants.

Remark: Problem with convex functional but non-convex constraints.

Static Osseen-Frank's theory for nematic LC

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$): domain filled by the LC, with boundary $\partial\Omega$
- Equilibrium states: minimum of a free energy
- **Oseen-Frank** free energy derive to the Static (Minimization) problem:

$$\min_{|\boldsymbol{d}|=1} E_{ela}(\boldsymbol{d})$$

where E_{ela} is the **elastic** energy functional:

$$E_{ela}(\boldsymbol{d}) = \int_\Omega \left(\frac{k_1}{2}(\nabla \cdot \boldsymbol{d})^2 + \frac{k_2}{2}(\boldsymbol{d} \cdot (\nabla \times \boldsymbol{d}))^2 + \frac{k_3}{2}|\boldsymbol{d} \times (\nabla \times \boldsymbol{d})|^2 \right)$$

$k_i > 0$ elastic constants.

Remark: Problem with convex functional but non-convex constraints.

Smectic-A LC

- **Smectic LC**: the molecules have an orientational order vector $\boldsymbol{d}$ and a positional order (layer structure of normal vector $\boldsymbol{n}$). Assume:
  - $\exists$ a potential function φ s.t. layers $=$ level sets of φ :

$$\nabla \times \boldsymbol{n} = 0 \quad \Longleftrightarrow \quad \boldsymbol{n} = \nabla\varphi$$

  - Incompressibility of the layers: $|\boldsymbol{n}| = 1$.
- **Smectic-A LC**: Assume $\boldsymbol{d} = \boldsymbol{n}$.
- Since $\boldsymbol{d} = \boldsymbol{n}$ and $\boldsymbol{n} = \nabla\varphi$, the elastic energy can be rewritten (reduced form) as

$$E_{ela} = \frac{1}{2}\int_\Omega |\nabla \cdot \boldsymbol{n}|^2 = \frac{1}{2}\int_\Omega |\Delta\varphi|^2$$

Smectic-A LC

- **Smectic LC**: the molecules have an orientational order vector $\boldsymbol{d}$ and a positional order (layer structure of normal vector $\boldsymbol{n}$). Assume:
  - $\exists$ a potential function φ s.t. layers $=$ level sets of φ :

$$\nabla \times \boldsymbol{n} = 0 \quad \Longleftrightarrow \quad \boldsymbol{n} = \nabla\varphi$$

  - Incompressibility of the layers: $|\boldsymbol{n}| = 1$.
- **Smectic-A LC**: Assume $\boldsymbol{d} = \boldsymbol{n}$.
- Since $\boldsymbol{d} = \boldsymbol{n}$ and $\boldsymbol{n} = \nabla\varphi$, the elastic energy can be rewritten (reduced form) as

$$E_{ela} = \frac{1}{2}\int_\Omega |\nabla \cdot \boldsymbol{n}|^2 = \frac{1}{2}\int_\Omega |\Delta\varphi|^2$$

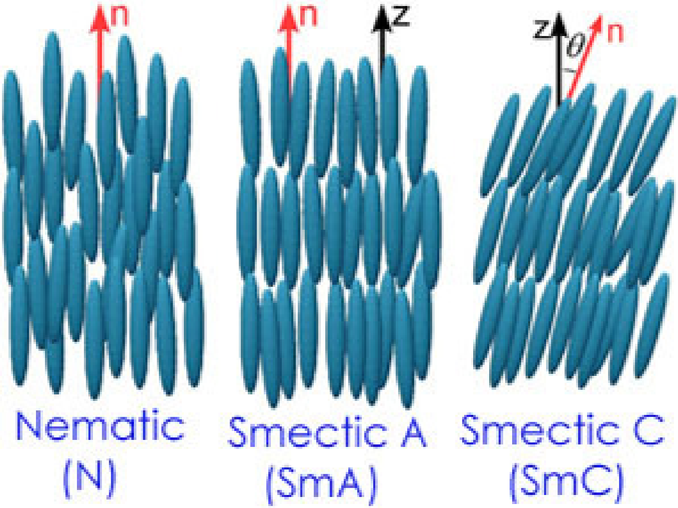
Smectic-A LC

- **Smectic LC**: the molecules have an orientational order vector $\boldsymbol{d}$ and a positional order (layer structure of normal vector $\boldsymbol{n}$). Assume:
  - $\exists$ a potential function φ s.t. layers $=$ level sets of φ :

$$\nabla \times \boldsymbol{n} = 0 \quad \Longleftrightarrow \quad \boldsymbol{n} = \nabla \varphi$$

  - Incompressibility of the layers: $|\boldsymbol{n}| = 1$.
- **Smectic-A LC**: Assume $\boldsymbol{d} = \boldsymbol{n}$.
- Since $\boldsymbol{d} = \boldsymbol{n}$ and $\boldsymbol{n} = \nabla\varphi$, the elastic energy can be rewritten (reduced form) as

$$E_{ela} = \frac{1}{2}\int_\Omega |\nabla \cdot \boldsymbol{n}|^2 = \frac{1}{2}\int_\Omega |\Delta\varphi|^2$$


n
n
z
z
θ
n
Nematic
(N)
Smectic A
(SmA)
Smectic C
(SmC)

A static Smectic-A LC problem

- Regularized Oseen-Frank energy (by penalization of **Ginzburg-Landau** type):

$$E_e := \int_\Omega \left(\frac{1}{2}(\Delta\varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla\varphi) \right), \quad F(\boldsymbol{n}) = \frac{1}{4}(|\boldsymbol{n}|^2 - 1)^2$$

Static (Minimization) problem:

$$\min_{\varphi} E_e$$

Remark: Problem without constraints but non-convex functional.

- Optimality system:

$$\left\langle \frac{\delta E_e(\varphi)}{\delta\varphi}, \overline{\varphi} \right\rangle = \int_\Omega \Delta\varphi\Delta\overline{\varphi} + \frac{1}{\varepsilon^2} \boldsymbol{f}(\nabla\varphi)\cdot\nabla\overline{\varphi} = 0 \qquad \forall\,\overline{\varphi}$$

where $\boldsymbol{f}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F(\boldsymbol{n}) = (|\boldsymbol{n}|^2 - 1)\boldsymbol{n}$.

A static Smectic-A LC problem

- Regularized Oseen-Frank energy (by penalization of **Ginzburg-Landau** type):

$$E_e := \int_\Omega \left(\frac{1}{2} (\Delta\varphi)^2 + \frac{1}{\varepsilon^2} F(\nabla\varphi) \right), \quad F(\boldsymbol{n}) = \frac{1}{4}(|\boldsymbol{n}|^2 - 1)^2$$

  Static (Minimization) problem:

$$\min_\varphi E_e$$

  Remark: Problem without constraints but non-convex functional.

- Optimality system:

$$\left\langle \frac{\delta E_e(\varphi)}{\delta\varphi}, \overline{\varphi} \right\rangle = \int_\Omega \Delta\varphi\Delta\overline{\varphi} + \frac{1}{\varepsilon^2} \boldsymbol{f}(\nabla\varphi)\cdot\nabla\overline{\varphi} = 0 \qquad \forall\, \overline{\varphi}$$

  where $\boldsymbol{f}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F(\boldsymbol{n}) = (|\boldsymbol{n}|^2 - 1)\boldsymbol{n}$.

$L^2(\Omega)$-Identification of the variational derivative

1. Assuming $\Delta\varphi|_{\partial\Omega} = 0$ (N1) or $\nabla\overline{\varphi}\cdot \boldsymbol{m}|_{\partial\Omega} = 0$ (D2) ($\boldsymbol{m}$ is the normal vector):

$$\left\langle \frac{\delta E_e(\varphi)}{\delta\varphi}, \overline{\varphi} \right\rangle = -\int_\Omega \boldsymbol{w}\cdot\nabla\overline{\varphi}, \quad \boldsymbol{w} := \nabla\Delta\varphi - \frac{1}{\varepsilon^2}\boldsymbol{f}(\nabla\varphi).$$

2. Assuming $\boldsymbol{w}\cdot\boldsymbol{m}|_{\partial\Omega} = 0$ (N2) or $\overline{\varphi}|_{\partial\Omega} = 0$ (D1):

$$\frac{\delta E_e}{\delta\varphi} \equiv \nabla\cdot\boldsymbol{w} = \Delta^2\varphi - \frac{1}{\varepsilon^2}\nabla\cdot\boldsymbol{f}(\nabla\varphi) \quad \text{in } \Omega.$$

$L^2(\Omega)$-Identification of the variational derivative

1. Assuming $\Delta\varphi|_{\partial\Omega} = 0$ (N1) or $\nabla\overline{\varphi}\cdot \boldsymbol{m}|_{\partial\Omega} = 0$ (D2) ($\boldsymbol{m}$ is the normal vector):

$$\left\langle \frac{\delta E_e(\varphi)}{\delta\varphi}, \overline{\varphi} \right\rangle = -\int_\Omega \boldsymbol{w}\cdot\nabla\overline{\varphi}, \quad \boldsymbol{w} := \nabla\Delta\varphi - \frac{1}{\varepsilon^2}\boldsymbol{f}(\nabla\varphi).$$

2. Assuming $\boldsymbol{w}\cdot\boldsymbol{m}|_{\partial\Omega} = 0$ (N2) or $\overline{\varphi}|_{\partial\Omega} = 0$ (D1):

$$\frac{\delta E_e}{\delta\varphi} \equiv \nabla\cdot\boldsymbol{w} = \Delta^2\varphi - \frac{1}{\varepsilon^2}\nabla\cdot\boldsymbol{f}(\nabla\varphi) \quad \text{in } \Omega.$$

Adimissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \boldsymbol{w} = 0$

$[D1 - D2]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2,$

$[D1 - N1]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \Delta\varphi|_{\partial\Omega} = 0,$

$[D2 - N2]$ $\quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0,$

$[N1 - N2]$ $\quad \Delta\varphi|_{\partial\Omega} = 0, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0.$

Non admissible B.C.: [D1-N2] and [D2-N1]

Adimissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \boldsymbol{w} = 0$

$[D1 - D2]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2,$

$[D1 - N1]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \Delta\varphi|_{\partial\Omega} = 0,$

$[D2 - N2]$ $\quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0,$

$[N1 - N2]$ $\quad \Delta\varphi|_{\partial\Omega} = 0, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0.$

Non admissible B.C.: [D1-N2] and [D2-N1]

Adimissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \boldsymbol{w} = 0$

$[D1 - D2]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2,$

$[D1 - N1]$ $\quad \varphi|_{\partial\Omega} = \varphi_1, \quad \Delta\varphi|_{\partial\Omega} = 0,$

$[D2 - N2]$ $\quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0,$

$[N1 - N2]$ $\quad \Delta\varphi|_{\partial\Omega} = 0, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0.$

Non admissible B.C.: [D1-N2] and [D2-N1]

Adimissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \boldsymbol{w} = 0$

$[D1 - D2]$ $\varphi|_{\partial\Omega} = \varphi_1, \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2,$

$[D1 - N1]$ $\varphi|_{\partial\Omega} = \varphi_1, \quad \Delta\varphi|_{\partial\Omega} = 0,$

$[D2 - N2]$ $\nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0,$

$[N1 - N2]$ $\Delta\varphi|_{\partial\Omega} = 0, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0.$

Non admissible B.C.: [D1-N2] and [D2-N1]

Adimissible B.C. to fourth-order Euler-Lagrange $\nabla \cdot \boldsymbol{w} = 0$

$$[D1 - D2] \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2,$$

$$[D1 - N1] \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \Delta\varphi|_{\partial\Omega} = 0,$$

$$[D2 - N2] \quad \nabla\varphi \cdot \boldsymbol{m}|_{\partial\Omega} = \varphi_2, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0,$$

$$[N1 - N2] \quad \Delta\varphi|_{\partial\Omega} = 0, \quad \boldsymbol{w} \cdot \boldsymbol{m}|_{\partial\Omega} = 0.$$

Non admissible B.C.: [D1-N2] and [D2-N1]

Smectic-A Liquid Crystal Model [E, ARMA'97]

- $\Omega \subset \mathbb{R}^3$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.
- **Linear momentum**, $(\boldsymbol{u}, p)$-system in Q:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \nabla\cdot(\sigma^d + \lambda\,\sigma^e) = 0, \quad \nabla\cdot\boldsymbol{u} = 0,$$

  - $\sigma^d = \sigma^d(D(\boldsymbol{u}), \boldsymbol{n})$: dissipative (symmetric) stress tensor:

$$\sigma^d = \mu_1(\boldsymbol{n}^t D\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n} + \mu_4 D + \mu_5(D\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D\boldsymbol{n}), \quad D = \frac{\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t}{2}$$

  - $\sigma^e = \sigma^e(\varphi)$: elastic (non-symmetric) stress tensor, s.t. $\nabla\cdot\sigma^e = \frac{\delta E_e}{\delta\varphi}\nabla\varphi + \nabla E_e$
- **Angular momentum**, φ-equation in Q (of Allen-Cahn's type):

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi + \gamma\frac{\delta E_e}{\delta\varphi} = 0,$$

$\gamma > 0$ constant (elastic relaxation time).

Smectic-A Liquid Crystal Model [E, ARMA'97]

- $\Omega \subset \mathbb{R}^3$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.
- **Linear momentum**, $(\boldsymbol{u}, p)$-system in Q:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \nabla \cdot (\sigma^d + \lambda \sigma^e) = 0, \quad \nabla \cdot \boldsymbol{u} = 0,$$

  - $\sigma^d = \sigma^d(D(\boldsymbol{u}), \boldsymbol{n})$: dissipative (symmetric) stress tensor:

$$\sigma^d = \mu_1(\boldsymbol{n}^t D\boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n} + \mu_4 D + \mu_5(D\boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes D\boldsymbol{n}), \quad D = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t}{2}$$

  - $\sigma^e = \sigma^e(\varphi)$: elastic (non-symmetric) stress tensor, s.t. $\nabla \cdot \sigma^e = \dfrac{\delta E_e}{\delta \varphi} \nabla \varphi + \nabla E_e$

- **Angular momentum**, φ-equation in Q (of Allen-Cahn's type):

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi + \gamma \frac{\delta E_e}{\delta \varphi} = 0,$$

$\gamma > 0$ constant (elastic relaxation time).

Smectic-A Liquid Crystal Model [E, ARMA'97]

- $\Omega \subset \mathbb{R}^3$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.
- **Linear momentum**, $(\boldsymbol{u}, p)$-system in Q:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \nabla \cdot (\sigma^d + \lambda\, \sigma^e) = 0, \quad \nabla \cdot \boldsymbol{u} = 0,$$

  - $\sigma^d = \sigma^d(D(\boldsymbol{u}), \boldsymbol{n})$: dissipative (symmetric) stress tensor:

$$\sigma^d = \mu_1(\boldsymbol{n}^t D\boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n} + \mu_4 D + \mu_5(D\boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes D\boldsymbol{n}), \quad D = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t}{2}$$

  - $\sigma^e = \sigma^e(\varphi)$: elastic (non-symmetric) stress tensor, s.t. $\nabla \cdot \sigma^e = \frac{\delta E_e}{\delta \varphi}\nabla\varphi + \nabla E_e$
- **Angular momentum**, φ-equation in Q (of Allen-Cahn's type):

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla\varphi + \gamma \frac{\delta E_e}{\delta \varphi} = 0,$$

$\gamma > 0$ constant (elastic relaxation time).

Smectic-A Liquid Crystal Model [E, ARMA'96]

- Initial-boundary problem (with Dirichlet [D1-D2] B.C. for φ) in Q:

$$
(P) \qquad \left\{
\begin{array}{rcll}
\dfrac{D\boldsymbol{u}}{Dt} - \nabla\cdot\sigma^d + \nabla\widetilde{p} - \lambda\dfrac{\delta E_e}{\delta\varphi}\nabla\varphi & = & \mathbf{0}, & \qquad \boldsymbol{u} \\
\nabla\cdot\boldsymbol{u} & = & 0, & \qquad p \\
\dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi + \gamma\dfrac{\delta E_e}{\delta\varphi} & = & 0, & \qquad \dfrac{\delta E_e}{\delta\varphi} \\
\Delta^2\varphi - \dfrac{1}{\varepsilon^2}\nabla\cdot\boldsymbol{f}(\nabla\varphi) & = & \dfrac{\delta E_e}{\delta\varphi}, & \qquad \dfrac{\partial\varphi}{\partial t} \\
\boldsymbol{u}|_\Sigma = 0, \quad \varphi|_\Sigma = \varphi_1, \quad \partial_n\varphi|_\Sigma & = & \varphi_2, & \\
\boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, \quad \varphi_{|t=0} & = & \varphi_0 &
\end{array}
\right.
$$

- $\widetilde{p} = p + \lambda E_e$ (Lagrange multiplier $\sim$ constraint $\nabla\cdot\boldsymbol{u} = 0$)
- Other B.C. are possible. For instance, $\boldsymbol{w}\cdot\boldsymbol{m}|_{\partial\Omega} = 0$ implies the conservation property $\dfrac{d}{dt}\displaystyle\int_\Omega \varphi = 0$.

Dissipative Energy law and weak solutions

Testing by $\boldsymbol{u}, p, \dfrac{\delta E_e}{\delta\varphi}, \dfrac{\partial\varphi}{\partial t}$:

$$(EL) \qquad \frac{d}{dt}\Big(E_{kin}(\boldsymbol{u}) + \lambda\, E_e(\varphi)\Big) + \int_\Omega \sigma^d : D + \lambda\gamma \int_\Omega \left|\frac{\delta E_e}{\delta\varphi}\right|^2 = 0$$

where

$$E_{kin}(\boldsymbol{u}) := \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2, \quad E_e := \int_\Omega \left(\frac{1}{2}(\Delta\varphi)^2 + \frac{1}{\varepsilon^2}F(\nabla\varphi)\right),$$

$$\sigma^d : D = \mu_1(\boldsymbol{n}^t D\boldsymbol{n})^2 + \mu_4|D|^2 + \mu_5|D\boldsymbol{n}|^2 \geq \mu_4|D|^2.$$

Weak solutions (variational $\sim (\boldsymbol{u}, p)$, point-wise $\sim \varphi$):

$$\boldsymbol{u} \in L^\infty(0,+\infty; \boldsymbol{L}^2(\Omega)) \cap L^2(0,+\infty; \boldsymbol{H}^1(\Omega)), \quad \frac{\delta E_e}{\delta\varphi} \in L^2(0,+\infty; L^2(\Omega)).$$

$$\varphi \in L^\infty(0,+\infty; H^2(\Omega)) \cap L^2_{loc}(0,+\infty; H^4(\Omega)), \quad \frac{\partial\varphi}{\partial t} \in L^2(0,+\infty; L^2(\Omega)).$$

Dissipative Energy law and weak solutions

Testing by $\boldsymbol{u}, p, \dfrac{\delta E_e}{\delta \varphi}, \dfrac{\partial \varphi}{\partial t}$:

$$(EL) \qquad \frac{d}{dt}\Big(E_{kin}(\boldsymbol{u}) + \lambda\, E_e(\varphi)\Big) + \int_\Omega \sigma^d : D + \lambda\gamma \int_\Omega \left|\frac{\delta E_e}{\delta \varphi}\right|^2 = 0$$

where

$$E_{kin}(\boldsymbol{u}) := \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2, \quad E_e := \int_\Omega \left(\frac{1}{2}(\Delta\varphi)^2 + \frac{1}{\varepsilon^2}F(\nabla\varphi)\right),$$

$$\sigma^d : D = \mu_1(\boldsymbol{n}^t D\boldsymbol{n})^2 + \mu_4|D|^2 + \mu_5|D\boldsymbol{n}|^2 \geq \mu_4|D|^2.$$

Weak solutions (variational $\sim (\boldsymbol{u}, p)$, point-wise $\sim \varphi$):

$$\mathbf{u} \in L^\infty(0,+\infty; \boldsymbol{L}^2(\Omega)) \cap L^2(0,+\infty; \boldsymbol{H}^1(\Omega)), \quad \frac{\delta E_e}{\delta\varphi} \in L^2(0,+\infty; L^2(\Omega)).$$

$$\varphi \in L^\infty(0,+\infty; H^2(\Omega)) \cap L^2_{loc}(0,+\infty; H^4(\Omega)), \quad \frac{\partial\varphi}{\partial t} \in L^2(0,+\infty; L^2(\Omega)).$$

Local strong estimates and strong solutions

- Combining adequately $\boldsymbol{u}$-system by $A\boldsymbol{u} + \partial_t \boldsymbol{u}$ and $\partial_t(\varphi\text{-eq})$ by $\partial_t\varphi$:

$$\Phi' + \Psi \le C(\Phi^3 + 1),$$

$$\Phi(t) := \|\boldsymbol{u}\|^2_{H^1} + \|\partial_t\varphi\|^2_{L^2}, \qquad \Psi(t) := \|\boldsymbol{u}\|^2_{H^2} + \|\partial_t \boldsymbol{u}\|^2_{L^2} + \|\partial_t\varphi\|^2_{H^2}$$

- **Strong (or regular) solutions** (full point-wise):

$$\boldsymbol{u} \in L^\infty(0,+\infty; \boldsymbol{H}^1(\Omega)) \cap L^2_{loc}(0,+\infty; \boldsymbol{H}^2(\Omega)), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^2_{loc}(0,+\infty; \boldsymbol{L}^2(\Omega)),$$

$$\frac{\partial\varphi}{\partial t} \in L^\infty(0,+\infty; L^2(\Omega)) \cap L^2_{loc}(0,+\infty; H^2(\Omega)).$$

$$\varphi \in L^\infty(0,+\infty; H^4) \cap L^2_{loc}(0,+\infty; H^6),$$

- Combining adequately $\boldsymbol{u}$-system by $A\boldsymbol{u} + \partial_t \boldsymbol{u}$ and $\partial_t(\varphi\text{-eq})$ by $\partial_t \varphi$:

$$\Phi' + \Psi \leq C(\Phi^3 + 1),$$

$$\Phi(t) := \|\boldsymbol{u}\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2, \qquad \Psi(t) := \|\boldsymbol{u}\|_{H^2}^2 + \|\partial_t \boldsymbol{u}\|_{L^2}^2 + \|\partial_t \varphi\|_{H^2}^2$$

- **Strong (or regular) solutions** (full point-wise):

$$\boldsymbol{u} \in L^\infty(0,+\infty; \boldsymbol{H}^1(\Omega)) \cap L^2_{loc}(0,+\infty; \boldsymbol{H}^2(\Omega)), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^2_{loc}(0,+\infty; \boldsymbol{L}^2(\Omega)),$$

$$\frac{\partial \varphi}{\partial t} \in L^\infty(0,+\infty; L^2(\Omega)) \cap L^2_{loc}(0,+\infty; H^2(\Omega)).$$

$$\varphi \in L^\infty(0,+\infty; H^4) \cap L^2_{loc}(0,+\infty; H^6),$$

Previous results

- **[Liu, DCDS'00]** Time-independent [D1-D2] B.C.:

  Existence of global in time weak solutions.

  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 >>$), and its ω-limit is a nonempty set furnished by "equilibrium points". Global minimizers of the elastic energy E_e are stable.

- **[Climent-Ezquerra & GG, CPAA'10]** Extension for time-dependent [D1-D2] B.C.

  Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 >>$.

- **[Segatti & Wu, SIMA'11]** Long-time behaviour with periodic B.C.:

  Finite dimensional attractor in $2D$ domains.

  Convergence to a single equilibrium (and convergence rate)

  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\dfrac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$.

  Moreover, local minimizers of the elastic energy E_e are stable.

  RK: It may exists a "continuum" of equilibrium solutions with the same energy.

Previous results

- **[Liu, DCDS'00]** Time-independent [D1-D2] B.C.:

  Existence of global in time weak solutions.

  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 >>$), and its ω-limit is a nonempty set furnished by "equilibrium points". Global minimizers of the elastic energy E_e are stable.

- **[Climent-Ezquerra & GG, CPAA'10]** Extension for time-dependent [D1-D2] B.C.

  Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 >>$.

- **[Segatti & Wu, SIMA'11]** Long-time behaviour with periodic B.C.:

  Finite dimensional attractor in $2D$ domains.

  Convergence to a single equilibrium (and convergence rate)

  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\dfrac{\delta E_e(\varphi_\infty)}{\delta\varphi} = 0$.

  Moreover, local minimizers of the elastic energy E_e are stable.

  RK: It may exists a "continuum" of equilibrium solutions with the same energy.

Previous results

- **[Liu, DCDS'00]** Time-independent [D1-D2] B.C.:

  Existence of global in time weak solutions.

  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 >>$), and its ω-limit is a nonempty set furnished by "equilibrium points". Global minimizers of the elastic energy E_e are stable.

- **[Climent-Ezquerra & GG, CPAA'10]** Extension for time-dependent [D1-D2] B.C.

  Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 >>$.

- **[Segatti & Wu, SIMA'11]** Long-time behaviour with periodic B.C.:

  Finite dimensional attractor in $2D$ domains.

  Convergence to a single equilibrium (and convergence rate)

  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\dfrac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$.

  Moreover, local minimizers of the elastic energy E_e are stable.

  RK: It may exists a "continuum" of equilibrium solutions with the same energy.

- **[Liu, DCDS'00]** Time-independent [D1-D2] B.C.:

  Existence of global in time weak solutions.

  Existence (and uniqueness) of global in time regular solutions for large viscosity ($\mu_4 >>$), and its ω-limit is a nonempty set furnished by "equilibrium points". Global minimizers of the elastic energy E_e are stable.

- **[Climent-Ezquerra & GG, CPAA'10]** Extension for time-dependent [D1-D2] B.C.

  Moreover, Existence of time-periodic weak solutions and regularity for $\mu_4 >>$.

- **[Segatti & Wu, SIMA'11]** Long-time behaviour with periodic B.C.:

  Finite dimensional attractor in $2D$ domains.

  Convergence to a single equilibrium (and convergence rate)

  In particular, $\varphi(t) \to \varphi_\infty$ as $t \to \infty$, with $\dfrac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$.

  Moreover, local minimizers of the elastic energy E_e are stable.

  RK: It may exists a "continuum" of equilibrium solutions with the same energy.

Main result

Let S be the set of equilibrium points:

$$S = \{(0, \overline{\varphi}) : \ \overline{\varphi} \in H^4(\Omega), \ \Delta^2 \overline{\varphi} - \nabla \cdot \frac{1}{\varepsilon^2} \boldsymbol{f}(\overline{\varphi}) = 0, \ \overline{\varphi}|_{\partial\Omega} = \varphi_1, \ \partial_{\mathrm{n}} \overline{\varphi}|_{\partial\Omega} = \varphi_2\}.$$

and the ω-limit set of a global weak solution:

$$\omega(\boldsymbol{u}, \varphi) = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{V} \times \boldsymbol{H}^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times \boldsymbol{H}^4\}.$$

Theorem

- $\omega(\boldsymbol{u}, \varphi) \subset S$
- $\omega(\boldsymbol{u}, \varphi) = \{(0, \varphi_\infty)\}$

Goal: Long-time behavior with admissible B.C.

Generic situation: Let $E(t), D(t) \geq 0$ satisfying the **"weak estimates"**:

$$(WE) \qquad E'(t) + D(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty).$$

Then, $E \in L^\infty(0, +\infty)$ and $E(t) \searrow E_\infty \geq 0$. Moreover, integrating (WE), one has $D \in L^1(0, +\infty)$.

Lemma ([Climent-Ezquerra,Rodriguez-Bellido & GG, JBC'10])

Let $D(t) \geq 0, D \in L^1(0, +\infty)$ satisfying the **"strong estimate"***:*

$$(SE) \qquad D'(t) \leq K(D(t)^3 + 1) \qquad (K > 0).$$

Then, there exists $T^* \geq 0$ (large enough) such that $D \in L^\infty(T^*, +\infty)$ and

$$\exists \lim_{t \to +\infty} D(t) = 0.$$

Goal: Long-time behavior with admissible B.C.

Generic situation: Let $E(t), D(t) \geq 0$ satisfying the **"weak estimates"**:

$$(WE) \qquad E'(t) + D(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty).$$

Then, $E \in L^\infty(0, +\infty)$ and $E(t) \searrow E_\infty \geq 0$. Moreover, integrating (WE), one has $D \in L^1(0, +\infty)$.

Lemma ([Climent-Ezquerra,Rodriguez-Bellido & GG, JBC'10])

Let $D(t) \geq 0$, $D \in L^1(0, +\infty)$ satisfying the **"strong estimate"***:*

$$(SE) \qquad D'(t) \leq K(D(t)^3 + 1) \qquad (K > 0).$$

Then, there exists $T^* \geq 0$ (large enough) such that $D \in L^\infty(T^*, +\infty)$ and

$$\exists \lim_{t \to +\infty} D(t) = 0.$$

Extension for sequences

Goal: Convergence uniformly with respect to the Galerkin sequence's index.

Theorem ([Climent-Ezquerra & GG, NA-TMA'14])

Let $E^m(t), D^m(t) \geq 0$, *satisfying*

(*WE*) $$(E^m)'(t) + D^m(t) \leq 0,$$

(*SE*) $$(D^m)'(t) \leq K(D^m(t)^3 + 1) \qquad K > 0 \text{ independent of } m.$$

Then, $\forall\, \varepsilon < 1, \exists\, T^* = T^*(\varepsilon) \geq 0$ *(large enough), independent of m, such that*

$$\|D^m\|_{L^\infty(T^*,+\infty)} \leq \varepsilon.$$

Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and
$$D(t) := \|\boldsymbol{u}(t)\|_{H^1}^2 + \left\| \frac{\delta E_e}{\delta \varphi} \right\|_{L^2}^2 \sim \|\boldsymbol{u}\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2$$
2. Existence of weak solutions $(\boldsymbol{u}, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T^*_{reg}, +\infty)$ for some $T^*_{reg} > 0$.
3. $E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\dfrac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:
$$\omega = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$$
Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
   - $\omega \neq \emptyset$.
   - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an “equilibrium point”).
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and
$$D(t) := \|\boldsymbol{u}(t)\|_{H^1}^2 + \left\| \frac{\delta E_e}{\delta\varphi} \right\|_{L^2}^2 \sim \|\boldsymbol{u}\|_{H^1}^2 + \|\partial_t\varphi\|_{L^2}^2$$
2. Existence of weak solutions $(\boldsymbol{u}, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T_{reg}^*, +\infty)$ for some $T_{reg}^* > 0$.
3. $E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}_0^1$ and $\dfrac{\delta E_e(\varphi(t))}{\delta\varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:
$$\omega = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists\{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$$
Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
    - $\omega \neq \emptyset$.
    - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta\varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an "equilibrium point").
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and
$D(t) := \|\boldsymbol{u}(t)\|^2_{H^1} + \left\| \frac{\delta E_e}{\delta \varphi} \right\|^2_{L^2} \sim \|\boldsymbol{u}\|^2_{H^1} + \|\partial_t \varphi\|^2_{L^2}$
2. Existence of weak solutions $(\boldsymbol{u}, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T^*_{reg}, +\infty)$ for some $T^*_{reg} > 0$.
3. $E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\dfrac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:
$\omega = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$
Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
   - $\omega \neq \emptyset$.
   - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an “equilibrium point”).
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and $D(t) := \|\boldsymbol{u}(t)\|_{H^1}^2 + \left\| \dfrac{\delta E_e}{\delta \varphi} \right\|_{L^2}^2 \sim \|\boldsymbol{u}\|_{H^1}^2 + \|\partial_t \varphi\|_{L^2}^2$
2. Existence of weak solutions $(\boldsymbol{u}, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T^*_{reg}, +\infty)$ for some $T^*_{reg} > 0$.
3. $E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\dfrac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:

   $\omega = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$

   Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:

   - $\omega \neq \emptyset$.
   - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an "equilibrium point").
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and
$D(t) := \|\boldsymbol{u}(t)\|^2_{H^1} + \left\|\dfrac{\delta E_e}{\delta\varphi}\right\|^2_{L^2} \sim \|\boldsymbol{u}\|^2_{H^1} + \|\partial_t\varphi\|^2_{L^2}$
2. Existence of weak solutions $(\boldsymbol{u},\varphi)$ in $(0,+\infty)$ which are strong solutions in $(T^*_{reg},+\infty)$ for some $T^*_{reg} > 0$.
3. $E(\boldsymbol{u}(t),\varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\dfrac{\delta E_e(\varphi(t))}{\delta\varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:
$\omega = \{(\boldsymbol{u}_\infty,\varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists\{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n),\varphi(t_n)) \to (\boldsymbol{u}_\infty,\varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$
Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
    - $\omega \neq \emptyset$.
    - If $(0,\varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta\varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an “equilibrium point”).
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

Main steps to prove convergence to a single steady equilibrium

1. To obtain (WE) and (SE) for $E(t) = E_{kin}(\boldsymbol{u}(t)) + \lambda\, E_e(\varphi(t))$ and
$D(t) := \|\boldsymbol{u}(t)\|^2_{H^1} + \left\| \frac{\delta E_e}{\delta \varphi} \right\|^2_{L^2} \sim \|\boldsymbol{u}\|^2_{H^1} + \|\partial_t \varphi\|^2_{L^2}$
2. Existence of weak solutions $(\boldsymbol{u}, \varphi)$ in $(0, +\infty)$ which are strong solutions in $(T^*_{reg}, +\infty)$ for some $T^*_{reg} > 0$.
3. $E(\boldsymbol{u}(t), \varphi(t)) \searrow E_\infty$ in $\mathbb{R}$, $\boldsymbol{u}(t) \to 0$ in $\boldsymbol{H}^1_0$ and $\dfrac{\delta E_e(\varphi(t))}{\delta \varphi} \to 0$ in L^2 (as $t \uparrow +\infty$).
4. Recall the ω-limit set:
$\omega = \{(\boldsymbol{u}_\infty, \varphi_\infty) \in \boldsymbol{H}^1 \times H^4 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), \varphi(t_n)) \to (\boldsymbol{u}_\infty, \varphi_\infty) \text{ in } \boldsymbol{H}^1 \times H^4\}$
Then, $\boldsymbol{u}_\infty = 0$ and $E_e(\varphi_\infty) = E_\infty$. Moreover:
   - $\omega \neq \emptyset$.
   - If $(0, \varphi_\infty) \in \omega$ then $\frac{\delta E_e(\varphi_\infty)}{\delta \varphi} = 0$ in Ω, + B.C. for φ_∞ (i.e. φ_∞ is an "equilibrium point").
5. (Application of Lojasiewicz-Simon). $\varphi(t) \to \varphi_\infty$ in H^4 as $t \uparrow +\infty$.

Theorem ("Strong" Lojasiewicz-Simon inequality for smectic-A problems)

If $\overline{\varphi} \in \mathcal{S}$, then there exists $C > 0$, $\beta > 0$ and $\theta \in (0, 1/2)$ depending on $\overline{\varphi}$ such that for all $\varphi \in H^4$ with $\varphi|_{\partial\Omega} = \varphi_1, \partial_n\varphi|_{\partial\Omega} = \varphi_2$ and $\|\varphi - \overline{\varphi}\|_{H^4} \leq \beta$, it holds

$$|E_e(\varphi) - E_e(\overline{\varphi})|^{1-\theta} \leq C \left\| \frac{\delta E_e(\varphi)}{\delta\varphi} \right\|_{L^2}$$

Proof: Application abstract result **[S.Z. Huang, Mathematical Surveys and Monographs, vol. 126, AMS, 2006.]**

Corollary (**[Segatti,Wu, SIMA'11]**)

It's possible to relax the local hypothesis $\|\varphi - \overline{\varphi}\|_{H^4} \leq \beta$ by $\|\varphi - \overline{\varphi}\|_{H^3} \leq \tilde{\beta}$

Numerical Aproximation

Scheme: Finite Difference in time and Mixed Finite Element in space.

For simplicity,

- Uniform partition in time $t_n = n\,k$ of $[0, T]$ with $k = T/N$ (time step).
- inf-sup stable pair of FE for velocity-pressure.

Difficulties:

- Approximate $\varphi \in \boldsymbol{H}^4(\Omega)$ by only continuous Finite Elements.
- Strongly nonlinear coupling between fluids and phase-field models.
- Decreasing discrete energy law $\Rightarrow$ energy-stability.

Numerical Aproximation

Scheme: Finite Difference in time and Mixed Finite Element in space.

For simplicity,

- Uniform partition in time $t_n = n\,k$ of $[0, T]$ with $k = T/N$ (time step).
- inf-sup stable pair of FE for velocity-pressure.

Difficulties:

- Approximate $\varphi \in \boldsymbol{H}^4(\Omega)$ by only continuous Finite Elements.
- Strongly nonlinear coupling between fluids and phase-field models.
- Decreasing discrete energy law $\Rightarrow$ energy-stability.

Mixed second-order reformulation of (P)

By using the auxiliary variable $\psi = -\Delta\varphi$:

$$\begin{cases} \Big(\dfrac{D\boldsymbol{u}}{Dt}, \overline{\boldsymbol{u}}\Big) + \Big(\sigma^d, D(\overline{\boldsymbol{u}})\Big) - \Big(p, \nabla\cdot\overline{\boldsymbol{u}}\Big) + \dfrac{\lambda}{\gamma}\Big((\dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi)\nabla\varphi, \overline{\boldsymbol{u}}\Big) = 0, \\ \Big(\nabla\cdot\boldsymbol{u}, \overline{p}\Big) = 0, \\ \dfrac{\lambda}{\gamma}\Big(\dfrac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi, \overline{\varphi}\Big) + \lambda\Big(\nabla\psi, \nabla\overline{\varphi}\Big) + \dfrac{\lambda}{\varepsilon^2}\Big(\boldsymbol{f}(\nabla\varphi), \nabla\overline{\varphi}\Big) = 0 \\ \Big(\partial_t\psi, \bar{\psi}\Big) - \Big(\nabla\partial_t\varphi, \nabla\bar{\psi}\Big) = 0. \end{cases}$$

Energy law: Choosing $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\varphi}, \overline{\psi}) = (\boldsymbol{u}, p, \partial_t\varphi, \lambda\psi)$

$$\frac{d}{dt}E_{tot}\Big(\boldsymbol{u}, \psi, \nabla\varphi\Big) + \Big(\sigma^d, D(\boldsymbol{u})\Big) + \frac{\lambda}{\gamma}\|\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi\|_{L^2}^2 = 0,$$

where $E_{tot}(\boldsymbol{u}, \psi, \nabla\varphi) := E_{kin}(\boldsymbol{u}) + \lambda\Big(E_{ela}(\psi) + E_{pen}(\nabla\varphi)\Big)$,

$$E_{kin}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|_{L^2}^2, \quad E_{ela}(\psi) = \frac{1}{2}\|\psi\|_{L^2}^2, \quad E_{pen}(\nabla\varphi) = \frac{1}{\varepsilon^2}\int_\Omega F(\nabla\varphi).$$

Mixed second-order reformulation of (P)

By using the auxiliary variable $\psi = -\Delta\varphi$:

$$\begin{cases} \left(\frac{D\boldsymbol{u}}{Dt}, \overline{\boldsymbol{u}}\right) + \left(\sigma^d, D(\overline{\boldsymbol{u}})\right) - \left(p, \nabla\cdot\overline{\boldsymbol{u}}\right) + \frac{\lambda}{\gamma}\left(\left(\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi\right)\nabla\varphi, \overline{\boldsymbol{u}}\right) = 0, \\ \left(\nabla\cdot\boldsymbol{u}, \overline{p}\right) = 0, \\ \frac{\lambda}{\gamma}\left(\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi, \overline{\varphi}\right) + \lambda\left(\nabla\psi, \nabla\overline{\varphi}\right) + \frac{\lambda}{\varepsilon^2}\left(\boldsymbol{f}(\nabla\varphi), \nabla\overline{\varphi}\right) = 0 \\ \left(\partial_t\psi, \bar{\psi}\right) - \left(\nabla\partial_t\varphi, \nabla\bar{\psi}\right) = 0. \end{cases}$$

Energy law: Choosing $(\overline{\boldsymbol{u}}, \overline{p}, \overline{\varphi}, \overline{\psi}) = (\boldsymbol{u}, p, \partial_t\varphi, \lambda\psi)$

$$\frac{d}{dt}E_{tot}\left(\boldsymbol{u}, \psi, \nabla\varphi\right) + \left(\sigma^d, D(\boldsymbol{u})\right) + \frac{\lambda}{\gamma}\|\frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\nabla\varphi\|^2_{L^2} = 0,$$

where $E_{tot}(\boldsymbol{u}, \psi, \nabla\varphi) := E_{kin}(\boldsymbol{u}) + \lambda\left(E_{ela}(\psi) + E_{pen}(\nabla\varphi)\right)$,

$$E_{kin}(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2}, \quad E_{ela}(\psi) = \frac{1}{2}\|\psi\|^2_{L^2}, \quad E_{pen}(\nabla\varphi) = \frac{1}{\varepsilon^2}\int_\Omega F(\nabla\varphi).$$

Generic second order scheme

Given $(\boldsymbol{u}^n, \varphi^n, \psi^n)$, compute $(\boldsymbol{u}^{n+1}, p^{n+\frac{1}{2}}, \varphi^{n+1}, \psi^{n+1}) \in \boldsymbol{U}_h \times P_h \times \Phi_h \times \Psi_h$ such that for any $(\bar{\boldsymbol{u}}, \bar{p}, \bar{\varphi}, \bar{\psi}) \in \boldsymbol{U}_h \times P_h \times \Phi_h \times \Psi_h$:

$$
\begin{aligned}
&\Big(\delta_t \boldsymbol{u}^{n+1}, \bar{\boldsymbol{u}}\Big) + c\Big(\widetilde{\boldsymbol{u}}, \boldsymbol{u}^{n+\frac{1}{2}}, \bar{\boldsymbol{u}}\Big) + \Big(\sigma^d(D(\boldsymbol{u}^{n+\frac{1}{2}}), \nabla\widetilde{\varphi}), D(\bar{\boldsymbol{u}})\Big) \\
&\qquad -\Big(p^{n+\frac{1}{2}}, \nabla\cdot\bar{\boldsymbol{u}}\Big) + \frac{\lambda}{\gamma}\Big((\delta_t\varphi^{n+1} + \boldsymbol{u}^{n+\frac{1}{2}}\cdot\nabla\widetilde{\varphi})\nabla\widetilde{\varphi}, \bar{\boldsymbol{u}}\Big) = 0, \\
&\Big(\nabla\cdot\boldsymbol{u}^{n+\frac{1}{2}}, \bar{p}\Big) = 0, \\
&\frac{\lambda}{\gamma}\Big(\delta_t\varphi^{n+1} + \boldsymbol{u}^{n+\frac{1}{2}}\cdot\nabla\widetilde{\varphi}, \bar{\varphi}\Big) + \lambda\Big(\nabla\psi^{n+\frac{1}{2}}, \nabla\bar{\varphi}\Big) + \frac{\lambda}{\varepsilon^2}\Big(\boldsymbol{f}^k(\nabla\varphi^{n+1}, \nabla\varphi^n), \nabla\bar{\varphi}\Big) = 0, \\
&(\delta_t\psi^{n+1}, \bar{\psi}) - (\delta_t\nabla\varphi^{n+1}, \nabla\bar{\psi}) = 0
\end{aligned}
$$

where $\boldsymbol{u}^{n+\frac{1}{2}} = (\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)/2$ etc.

Unconditional energy stability

Lemma

The following discrete energy inequality holds:

$$\delta_t E_{tot}(\boldsymbol{u}^{n+1}, \nabla\varphi^{n+1}, w^{n+1}) + \mu_4 \|D(\boldsymbol{u}^{n+\frac{1}{2}})\|^2_{L^2(\Omega)}$$

$$+\frac{\lambda}{\gamma}\|\delta_t\varphi^{n+1} + \boldsymbol{u}^{n+\frac{1}{2}}\cdot\nabla\widetilde{\varphi}\|^2_{L^2(\Omega)} + \frac{\lambda}{\varepsilon^2} ND \leq 0,$$

where

$$ND = \int_\Omega \boldsymbol{f}^k(\nabla\varphi^{n+1}, \nabla\varphi^n)\cdot\delta_t\nabla\varphi^{n+1} - \delta_t\left(\int_\Omega F(\nabla\varphi^{n+1})\right).$$

Conclusions:

1. Regularity and uniqueness for large times
2. Convergence to single equilibrium steady solutions
3. Unconditional fully discrete energy-stable schemes

Open problems:

1. Long-time behaviour for other B.C.
2. Splitting energy-stable schemes
3. Numerical simulations

Conclusions and open problems

Conclusions:

1. Regularity and uniqueness for large times
2. Convergence to single equilibrium steady solutions
3. Unconditional fully discrete energy-stable schemes

Open problems:

1. Long-time behaviour for other B.C.
2. Splitting energy-stable schemes
3. Numerical simulations

THANK YOU FOR YOUR ATTENTION